Open mapping Theorem

Defn: Let B & B' be Banach spaces.

Let $S(x,r)$ be an open sphere with centre x and radius r in B.

Let $S'(x,r)$ be an open sphere with centre x and radius r in B'.

$S_r \to$ open sphere with centre at origin & radius r in B.

$S_r' \to$ open sphere with centre at origin & radius r in B'.

Then

$$S(x,r) = x + S_r \quad \& \quad S_r = r S_1$$

$S_1 \to$ open sphere with centre at origin & radius 1 in B.

If $y \in S(x,r)$, then

$$\|y - x\| < r$$

$$\Leftrightarrow \|z\| < r \quad \text{with } y - x = z$$

$$\Leftrightarrow y = x + z \quad \& \quad \underbrace{\|z\| < r}_{S_r}$$

$$\Leftrightarrow y \in x + S_r$$

and

$$S_r = \{x : \|x\| < r\}$$

$$= \left\{x : \frac{\|x\|}{r} < 1\right\}$$

$$= \{ry : \underbrace{\|y\| < 1}_{S_1}\}$$

$$= r S_1$$

Lemma: Let B & B' be Banach spaces and T a cont. L.T. of B onto B'. Then the image of each open sphere centred on the origin in B contains an open sphere centred on the origin in B'.

Note: This lemma will be used to prove open mapping theorem.

Open mapping Theorem:

Let B & B' be Banach spaces. If T is a continuous L.T. of B onto B' then T is an open mapping.

Pf → Given → $T: B \to B'$ cont. & onto.

To show that T is open, we shall show that for every open set G in B, its image $T[G]$ in B' is also open. For this, we shall show that for any $y \in T[G]$, $\exists$ an open sphere in B' contained in $T[G]$.

Now → let $y \in T[G]$ be arbitrary.

Then $y = T(x)$ for some $x \in G$

Since G is an open set in B, $\therefore$ $\exists$ an open sphere $S(x, r)$ in B centred at x s.t. $S(x, r) \subset G$.

We write $S(x,r)$ as $x+S_r$ i.e.

$$S(x,r) = x+S_r$$

where S_r is an open sphere in B centred at origin.

$$\therefore\ x+S_r \subset G \longrightarrow (1)$$

By above lemma, $\exists$ an open sphere S'_ϵ in B' centred at origin s.t.

$$S'_\epsilon \subset T[S_r]$$

$$\therefore\ y+S'_\epsilon \subset y+T[S_r] = T(x)+T[S_r]$$

as $y = T(x)$

$\Rightarrow$ $y+S'_\epsilon \subset T[x+S_r]$, $\because T$ is linear

$\Rightarrow$ $S'(y,\epsilon) \subset T[x+S_r]$, $\because y+S'_\epsilon = S'(y,\epsilon)$

$\Rightarrow$ $S'(y,\epsilon) \subset T[G]$, from (1).

Thus, we have shown that for any arbitrary point $y \in T[G]$, $\exists$ an open sphere in B' centred at y and contained in $T[G]$.

$\therefore\ T[G]$ is an open set.

(Proved)